The exam is 2 hours long.
No calculator, use of internet or use of AI are allowed.
Score 80 or higher to get 100% .

1. (10 points) Pay attention to clarity and readability.

1 Exercises

2. (3 points) Find the total differential of $w = (x + y)/(z - 3y)$.

Solution: The total differential dw of a function $w(x, y, z)$ is given by the formula:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

First, we compute the partial derivative with respect to x:

$$\frac{\partial w}{\partial x} = \frac{1}{z - 3y}.$$

Next, using the quotient rule, we compute the partial derivative with respect to y:

$$\frac{\partial w}{\partial y} = \frac{1 \cdot (z - 3y) - (x + y)(-3)}{(z - 3y)^2} = \frac{z - 3y + 3x + 3y}{(z - 3y)^2} = \frac{3x + z}{(z - 3y)^2}.$$

Finally, we compute the partial derivative with respect to z:

$$\frac{\partial w}{\partial z} = (x + y) \cdot \left(-(z - 3y)^{-2}\right) = -\frac{x + y}{(z - 3y)^2}.$$

Combining these partial components, the total differential is:

$$dw = \frac{1}{z - 3y}dx + \frac{3x + z}{(z - 3y)^2}dy - \frac{x + y}{(z - 3y)^2}dz.$$

3. (4 points) Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but that f is not differentiable at $(0, 0)$:

$$f(x, y) = \begin{cases} 1, & x = 0 \text{ or } y = 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution: First, we evaluate the existence of the partial derivatives at $(0, 0)$ using the formal limit definition. For $f_x(0, 0)$:

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}.$$

Since the point $(h, 0)$ lies directly on the x-axis where $y = 0$, we have $f(h, 0) = 1$. Additionally, $f(0, 0) = 1$. This yields:

$$f_x(0, 0) = \lim_{h\to 0} \frac{1 - 1}{h} = 0.$$

By symmetry, calculating the partial derivative with respect to y yields:

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{1-1}{k} = 0$$

Thus, both partial derivatives exist at the origin and are equal to 0.

Next, we check for differentiability. A fundamental theorem states that if a function is differentiable at a point, it must also be continuous at that point. Let us test the continuity of $f(x,y)$ at $(0,0)$ by approaching along the line $y = x$ (where $x \neq 0$):

$$\lim_{\substack{(x,y) \to (0,0) \\ y=x}} f(x,y) = \lim_{x \to 0} 0 = 0$$

However, the value of the function at the origin is $f(0,0) = 1$. Since the limit along this path ($\to 0$) does not match the actual functional value (1), $f(x,y)$ is discontinuous at $(0,0)$. Consequently, it cannot be differentiable at $(0,0)$.

4. (3 points) Find the directional derivative of f in the direction of $\boldsymbol{v}$ at the point P :

$$f(x,y) = x^3 - y^3, \quad P(4,3), \quad \boldsymbol{v} = \frac{\sqrt{2}}{2}(\boldsymbol{i} + \boldsymbol{j}).$$

Solution: First, we verify that $\boldsymbol{v}$ is a unit direction vector by calculating its magnitude:

$$\|\boldsymbol{v}\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

Next, we determine the gradient vector $\nabla f(x,y)$ by taking the first partial derivatives:

$$\nabla f(x,y) = \langle 3x^2, -3y^2 \rangle.$$

Evaluating this gradient vector at our specific point $P(4,3)$:

$$\nabla f(4,3) = \langle 3(4^2), -3(3^2) \rangle = \langle 48, -27 \rangle.$$

The directional derivative is computed as the dot product of the gradient and our unit vector $\boldsymbol{v}$:

$$D_{\boldsymbol{v}} f(4,3) = \nabla f(4,3) \cdot \boldsymbol{v} = \langle 48, -27 \rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = 48 \left(\frac{\sqrt{2}}{2}\right) - 27 \left(\frac{\sqrt{2}}{2}\right) = \frac{21\sqrt{2}}{2}.$$

5. (6 points) Let $\boldsymbol{F}(x,y) = (x^3 + e^y)\boldsymbol{i} + (xe^y - 6)\boldsymbol{j}$.

   (a) Determine whether $\boldsymbol{F}$ is conservative, and if it is give a potential function.

Solution: Let $\boldsymbol{F}(x,y) = L(x,y)\boldsymbol{i} + M(x,y)\boldsymbol{j}$. Assuming the textbook notation typo where $x^e y$ is intended to be xe^y, we have $L(x,y) = x^3 + e^y$ and $M(x,y) = xe^y - 6$. We check for conservativeness on an open, simply connected domain by validating if $\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x}$:

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}(x^3 + e^y) = e^y, \quad \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(xe^y - 6) = e^y.$$

Since $\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x}$, the vector field $\boldsymbol{F}$ is conservative.
To construct the corresponding potential function $f(x, y)$ such that $\nabla f = \boldsymbol{F}$, we integrate L with respect to x:

$$f(x, y) = \int (x^3 + e^y)\, dx = \frac{1}{4}x^4 + xe^y + g(y).$$

To resolve the unknown function $g(y)$, we differentiate our expression with respect to y and set it equal to $M(x, y)$:

$$\frac{\partial f}{\partial y} = xe^y + g'(y) = xe^y - 6 \implies g'(y) = -6 \implies g(y) = -6y + C.$$

Choosing $C = 0$, a potential function is:

$$f(x, y) = \frac{1}{4}x^4 + xe^y - 6y.$$

(b) Let $\mathcal{C}$ be a closed loop on the triangle defined by the points $(0, 0)$, $(1, 0)$ and $(1, 1)$. Compute the following integral :

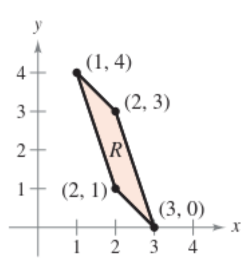
$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Solution: By the fundamental properties of conservative vector fields, the line integral of a conservative field along any closed path $\mathcal{C}$ depends only on the endpoints. Since a closed loop starts and ends at the exact same location, the circulation integral is identically zero:

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = 0.$$

6. (8 points) (a) Sketch the image S in the uv-plane of the region R in the xy-plane using the given transformation :

$$\begin{cases} x = & \frac{1}{2}(v - u), \\ y = & \frac{1}{2}(3u - v). \end{cases}$$



Solution: To map the boundary limits to the uv-plane, we solve the system of transformation equations for u and v. Adding the two formulas yields:

$$x + y = \frac{1}{2}v - \frac{1}{2}u + \frac{3}{2}u - \frac{1}{2}v = u \implies u = x + y.$$

Next, multiplying the first equation by 3 and adding it to the second yields:

$$3x + y = \frac{3}{2}v - \frac{3}{2}u + \frac{3}{2}u - \frac{1}{2}v = v \implies v = 3x + y.$$

For this classic problem layout, the parallelogram R in the xy-plane is bounded by the lines $x + y = 1$, $x + y = 2$, $3x + y = 0$, and $3x + y = 2$. Substituting our simplified expressions for u and v directly into these boundary equations defines the region S in the uv-plane as a simple rectangle:

$$1 \leq u \leq 2, \quad 0 \leq v \leq 2.$$

(b) Use the indicated change of variables to evaluate the double integral

$$\int_R \int (2y - x) dA.$$

Solution: First, we substitute our transformation equations to express the integrand in terms of u and v:

$$2y - x = 2\left(\frac{1}{2}(3u - v)\right) - \frac{1}{2}(v - u) = 3u - v - \frac{1}{2}v + \frac{1}{2}u = \frac{7}{2}u - \frac{3}{2}v.$$

Next, we evaluate the Jacobian determinant of the change of variables transformation:

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{2}.$$

The area element transformation requires the absolute value of the Jacobian: $dA = |J|\, du\, dv = \frac{1}{2}\, du\, dv$. Now we integrate over the rectangular region S:

$$\begin{aligned}
\iint_R (2y - x)\, dA &= \int_0^2 \int_1^2 \left(\frac{7}{2}u - \frac{3}{2}v\right) \cdot \frac{1}{2}\, du\, dv \\
&= \frac{1}{2}\int_0^2 \left[\frac{7}{4}u^2 - \frac{3}{2}vu\right]_1^2 dv \\
&= \frac{1}{2}\int_0^2 \left[(7 - 3v) - \left(\frac{7}{4} - \frac{3}{2}v\right)\right] dv \\
&= \frac{1}{2}\int_0^2 \left(\frac{21}{4} - \frac{3}{2}v\right) dv = \frac{1}{2}\left[\frac{21}{4}v - \frac{3}{4}v^2\right]_0^2 \\
&= \frac{1}{2}\left(\frac{21}{2} - 3\right) = \frac{1}{2}\left(\frac{15}{2}\right) = \frac{15}{4}.
\end{aligned}$$

7. (5 points) Find $\iint_S \boldsymbol{F} \cdot \boldsymbol{N} dS$ where $\boldsymbol{N}$ is the unit outward normal vector to $S : z = 1 - x - y$, first octant, and $\boldsymbol{F}(x, y, z) = 3z\boldsymbol{i} - 4\boldsymbol{j} + y\boldsymbol{k}$.

Solution: The surface S is a flat triangular plane segment satisfying $x + y + z = 1$. The outward unit normal pointing away from the origin solid in the first octant is given by $\boldsymbol{N} = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$. The standard surface area element dS for a plane modeled as $z = g(x, y) = 1 - x - y$ is:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \sqrt{1 + (-1)^2 + (-1)^2}\, dA = \sqrt{3}\, dA.$$

Combining these parameters yields $\boldsymbol{N}\, dS = \langle 1, 1, 1\rangle\, dA$. Next, we find the scalar dot product expression $\boldsymbol{F} \cdot \langle 1, 1, 1\rangle$:

$$\boldsymbol{F} \cdot \langle 1, 1, 1\rangle = 3z - 4 + y.$$

Substituting the plane's restriction equation $z = 1 - x - y$ isolates the expression into 2D variables:

$$3(1 - x - y) - 4 + y = -1 - 3x - 2y.$$

The region D in the xy-plane represents the projection of the first-octant boundary triangle, bounded by $x = 0$, $y = 0$, and $y = 1 - x$. Evaluating the double integral over D:

$$\begin{aligned}
\iint_S \boldsymbol{F} \cdot \boldsymbol{N}\, dS &= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y)\, dy\, dx \\
&= \int_0^1 \left[(-1 - 3x)y - y^2\right]_0^{1-x}\, dx \\
&= \int_0^1 \left((-1 - 3x)(1 - x) - (1 - x)^2\right)\, dx \\
&= \int_0^1 (1 - x)\left(-1 - 3x - (1 - x)\right)\, dx = \int_0^1 (1 - x)(-2 - 2x)\, dx \\
&= \int_0^1 -2(1 - x^2)\, dx = -2\left[x - \frac{x^3}{3}\right]_0^1 = -\frac{4}{3}.
\end{aligned}$$

8. (5 points) Let $\boldsymbol{r}(u, v) = u\boldsymbol{i} + v\boldsymbol{j} + \sqrt{uv}\boldsymbol{k}$. Find an equation of the tangent plane to the surface represented by $\boldsymbol{r}$ at the point $(1, 1, 1)$.

Solution: First, we determine the parameter values (u, v) that correspond to the point $(1, 1, 1)$. Since $x = u$ and $y = v$, we evaluate at $(u, v) = (1, 1)$.

Next, we compute the partial derivatives of the parameterized surface $\boldsymbol{r}(u, v) = \langle u, v, \sqrt{uv}\rangle$:

$$\boldsymbol{r}_u(u, v) = \left\langle 1, 0, \frac{1}{2}\sqrt{\frac{v}{u}}\right\rangle \implies \boldsymbol{r}_u(1, 1) = \left\langle 1, 0, \frac{1}{2}\right\rangle.$$

$$\boldsymbol{r}_v(u, v) = \left\langle 0, 1, \frac{1}{2}\sqrt{\frac{u}{v}}\right\rangle \implies \boldsymbol{r}_v(1, 1) = \left\langle 0, 1, \frac{1}{2}\right\rangle.$$

The normal vector $\boldsymbol{n}$ to the surface at this point is the cross product of these tangent vectors:

$$\boldsymbol{n} = \boldsymbol{r}_u(1, 1) \times \boldsymbol{r}_v(1, 1) = \left\langle 1, 0, \frac{1}{2}\right\rangle \times \left\langle 0, 1, \frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, -\frac{1}{2}, 1\right\rangle.$$

To simplify our plane equation, we can scale the normal vector by 2 to get $\boldsymbol{n} = \langle -1, -1, 2\rangle$.

Using the point-normal form of a plane, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$:

$$-1(x - 1) - 1(y - 1) + 2(z - 1) = 0 \implies -x - y + 2z = 0 \implies x + y - 2z = 0.$$

9. (12 points) The goal is to find the solid region V that minimizes the value of

$$\iiint_V (x^2 + y^2 + z^2 - 4)\, dV.$$

(a) Give the function $f(\rho, \theta, \phi)$ defined when we use a change of coordinates from Cartesian to spherical:

$$\iiint_V (x^2 + y^2 + z^2 - 4)\, dV = \int_{\rho=a}^{b} \int_{\phi=\phi_1}^{\phi_2} \int_{\theta=\theta_1}^{\theta_2} f(\rho, \theta, \phi)\, d\theta\, d\phi\, d\rho,$$

where $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$, $0 \leq \phi_1 \leq \phi_2 \leq \pi$, and $a \leq b$.

Solution: When switching to spherical coordinates, we substitute $x^2+y^2+z^2 = \rho^2$ and replace the volume differential dV with the spherical volume element $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. This yields:

$$\iiint_V (x^2 + y^2 + z^2 - 4)\, dV = \int_{\rho=a}^{b} \int_{\phi=\phi_1}^{\phi_2} \int_{\theta=\theta_1}^{\theta_2} (\rho^2 - 4) \cdot \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho.$$

Distributing ρ^2 through the radial term gives the function $f(\rho, \theta, \phi)$:

$$f(\rho, \theta, \phi) = (\rho^4 - 4\rho^2) \sin\phi.$$

(b) Prove

$$\iiint_V (x^2 + y^2 + z^2 - 4)\, dV = (\theta_2 - \theta_1)(\cos\phi_1 - \cos\phi_2) \left(\frac{b^5}{5} - \frac{4b^3}{3} - \frac{a^5}{5} + \frac{4a^3}{3} \right).$$

Solution: We evaluate the iterated integral found in part (a) using $f(\rho, \theta, \phi) = (\rho^4 - 4\rho^2) \sin\phi$:

$$\iiint_V (x^2 + y^2 + z^2 - 4)\, dV = \int_{\rho=a}^{b} (\rho^4 - 4\rho^2)\, d\rho \int_{\phi=\phi_1}^{\phi_2} \sin\phi\, d\phi \int_{\theta=\theta_1}^{\theta_2} d\theta.$$

Integrating each independent component:

$$\int_{\theta=\theta_1}^{\theta_2} d\theta = (\theta_2 - \theta_1)$$

$$\int_{\phi=\phi_1}^{\phi_2} \sin\phi\, d\phi = [-\cos\phi]_{\phi_1}^{\phi_2} = (\cos\phi_1 - \cos\phi_2)$$

$$\int_{\rho=a}^{b} (\rho^4 - 4\rho^2)\, d\rho = \left[\frac{\rho^5}{5} - \frac{4\rho^3}{3} \right]_a^b = \left(\frac{b^5}{5} - \frac{4b^3}{3} - \frac{a^5}{5} + \frac{4a^3}{3} \right).$$

Multiplying these evaluated components together yields the proven result:

$$\iiint_V (x^2 + y^2 + z^2 - 4)\, dV = (\theta_2 - \theta_1)(\cos\phi_1 - \cos\phi_2) \left(\frac{b^5}{5} - \frac{4b^3}{3} - \frac{a^5}{5} + \frac{4a^3}{3} \right).$$

(c) Let $g(a, b) = \frac{b^5}{5} - \frac{4b^3}{3} - \frac{a^5}{5} + \frac{4a^3}{3}$. We find the global minimum of the function $g(a, b)$ on the closed, constrained domain defined by the physical geometry of integration boundaries:

$$D = \{(a, b) \in \mathbb{R}^2 \mid 0 \leq a \leq b \leq 4\}.$$

i. Prove that if the interior domain is

$$\overset{\circ}{D} = \{ \ (a,b) \ / \ 0 < a < b < 4\},$$

then no critical points hold in $\overset{\circ}{D}$.

Solution: We compute the gradient vector $\nabla g(a,b)$ and set it equal to the zero vector $\mathbf{0}$:

$$\nabla g(a,b) = \left\langle \frac{\partial g}{\partial a}, \frac{\partial g}{\partial b} \right\rangle = \left\langle -a^4 + 4a^2, \, b^4 - 4b^2 \right\rangle = \langle 0, 0 \rangle$$

Solving the system of equations for the strictly positive interior ($0 < a < b$):

$$-a^2(a^2-4) = 0 \implies a = 2 \quad (\text{since } a > 0)$$
$$b^2(b^2-4) = 0 \implies b = 2 \quad (\text{since } b > 0)$$

The point $(2,2)$ lies exactly on the boundary line $a = b$, meaning there are no critical points strictly within the interior $\overset{\circ}{D}$.

ii. What theorem allows us to conclude that the minimum *must* exist on the boundary of D?

Solution: By the Extreme Value Theorem, because $g(a,b)$ is continuous on the closed and bounded region D, the global minimum must lie on the boundaries if none exist in the interior.

iii. By checking on the boundaries $(a,b) = (0,t)$ and $(a,b) = (t,t)$, prove that $(a,b) = (0,2)$ is a global minimum for g.

Solution: The boundary of the region D consists of two distinct components:

1. Boundary 1 ($a = 0$): Here, $b \geq 0$. Substituting $a = 0$ into $g(a,b)$ yields the single-variable function $g_1(b) = \frac{b^5}{5} - \frac{4b^3}{3}$. Differentiating gives $g_1'(b) = b^4 - 4b^2 = b^2(b-2)(b+2)$. For the domain $b \geq 0$, the only positive critical point is $b = 2$. Evaluating g at this point gives:

$$g(0,2) = \frac{2^5}{5} - \frac{4(2^3)}{3} = \frac{32}{5} - \frac{32}{3} = \frac{96-160}{15} = -\frac{64}{15}.$$

2. Boundary 2 ($a = b$): Substituting $a = b$ directly into $g(a,b)$ yields:

$$g(b,b) = \left(\frac{b^5}{5} - \frac{4b^3}{3}\right) - \left(\frac{b^5}{5} - \frac{4b^3}{3}\right) = 0.$$

Comparing the evaluated values found on the boundaries ($0 > -\frac{64}{15}$), the absolute lowest value achieved by the function is $-\frac{64}{15}$. Therefore, $(a,b) = (0,2)$ is the global minimum for g.

(d) Give without justification angles $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$ and $0 \leq \phi_1 \leq \phi_2 \leq \pi$ such that the angular multiplier is *maximized*:

$$\max_{\substack{0 \leq \theta_1 \leq \theta_2 \leq 2\pi \\ 0 \leq \phi_1 \leq \phi_2 \leq \pi}} \Big((\theta_2 - \theta_1)(\cos\phi_1 - \cos\phi_2) \Big).$$

Solution:

$$\theta_1 = 0, \quad \theta_2 = 2\pi, \quad \phi_1 = 0, \quad \phi_2 = \pi.$$

This generates the maximum scaling factor: $(2\pi - 0)(1 - (-1)) = 4\pi$.

(e) Give the absolute minimum value of the total integral, knowing :

$$\max_V \iiint_V x^2 + y^2 + z^2 - 4 \, dV = \left(\max_{\substack{0 \le \theta_1 \le \theta_2 \le 2\pi \\ 0 \le \phi_1 \le \phi_2 \le \pi}} \Big((\theta_2 - \theta_1)(\cos\phi_1 - \cos\phi_2) \Big) \right) g(0, 2).$$

Solution: To minimize the entire integral, we multiply our maximum positive angular scaling factor by the global minimum of the radial polynomial $g(a, b)$:

$$\text{Minimum Value} = 4\pi \cdot g(0, 2) = 4\pi \left(-\frac{64}{15} \right) = -\frac{256\pi}{15}$$

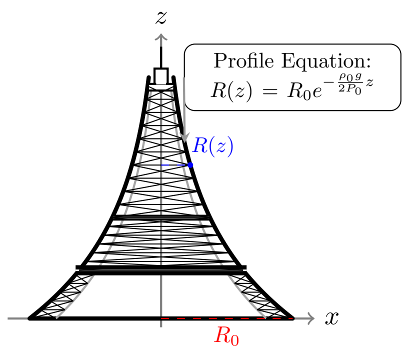
Thus, the absolute minimum value of the integral is $-\frac{256\pi}{15}$, which geometrically occurs over the full solid sphere of radius 2 centered at the origin.

2 Problem

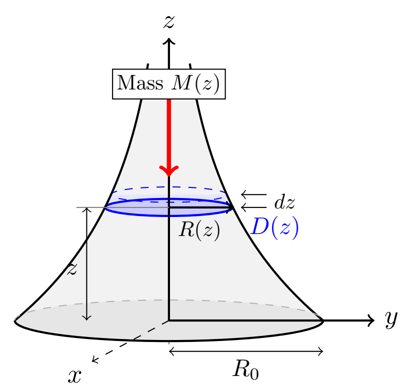
> The problem is partitioned into 3 independents parts.
> If a result from another part is needed, it is restated in the question.

Consider a rigid vertical tower modeled as a 3D solid of revolution E centered around the z-axis. The solid is bounded below by the ground plane $z = 0$, and extends infinitely upward ($z \to \infty$). At any height z, the horizontal cross-section $D(z)$ is a circular disk of radius $R(z)$. The bounding surface of the tower is given in cylindrical coordinates by $r = R(z)$.

Our architectural goal is to design the profile $R(z)$ such that the internal compressive pressure $P(z)$ caused by the weight of the material above it is perfectly uniform across every horizontal cross-section: $P(z) = P_0$ (a constant). The material has a uniform mass density ρ_0 and g is the acceleration due to gravity.



(a) View in the yz-plane of the structure.



(b) Continuous 3D model with slice elements.

Figure 1: Mathematical and structural representations of the tower.

10. (22 points) **Part 1 : Pressure equilibrium to construct a tower.**

(a) For a fixed z, state the region E_z inside the solid and above the cross section $D(z)$ in the cylindrical coordinates. You may use $\tilde{z}$ as the dummy coordinate for the vertical axis.

Solution:

$$E_z = \left\{ (r, \theta, \tilde{z}) \middle/ \begin{cases} z \leq \tilde{z} \leq +\infty \\ 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq R(\tilde{z}) \end{cases} \right\}.$$

(b) i. State the triple integral describing **total mass** $M(z)$ of the portion of the tower resting **above** a specific height z in cylindrical coordinates.

ii. Deduce from the triple integral that :

$$M(z) = \int_z^{+\infty} \rho_0 \pi [R(\tilde{z})]^2 \, d\tilde{z}.$$

Solution: The region E_z spans from altitude $\tilde{z} = z$ to $\tilde{z} \to \infty$. For a given height $\tilde{z}$, the radius ranges from $r = 0$ to $r = R(\tilde{z})$, and the azimuthal angle θ spans from 0 to 2π. Thus:

$$M(z) = \iiint_{E_z} \rho_0 \, dV = \int_z^{+\infty} \int_0^{2\pi} \int_0^{R(\tilde{z})} \rho_0 \cdot r \, dr \, d\theta \, d\tilde{z}.$$

Evaluating the inner parameters for r and θ:

$$\int_0^{2\pi} \int_0^{R(\tilde{z})} r \, dr \, d\theta = 2\pi \left[\frac{r^2}{2} \right]_0^{R(\tilde{z})} = \pi [R(\tilde{z})]^2.$$

Substituting this back into the outer integral yields:

$$M(z) = \int_z^{+\infty} \rho_0 \pi [R(\tilde{z})]^2 \, d\tilde{z}.$$

(c) Given that pressure is defined as $P_0 = \frac{\text{Force}}{\text{Area}} = \frac{M(z)g}{\pi[R(z)]^2}$, show that the boundary curve $R(z)$ must satisfy the master integral equation:

$$\pi [R(z)]^2 P_0 = \rho_0 g \int_z^{+\infty} \pi [R(\tilde{z})]^2 \, d\tilde{z}.$$

Solution: Substituting the evaluated single integral formula for $M(z)$ into the pressure condition equation:

$$P_0 = \frac{g \cdot M(z)}{\pi [R(z)]^2} \implies P_0 = \frac{g \int_z^{+\infty} \rho_0 \pi [R(\tilde{z})]^2 \, d\tilde{z}}{\pi [R(z)]^2}$$

Multiplying both sides by $\pi[R(z)]^2$ and isolating constants yields the required expression:

$$\pi [R(z)]^2 P_0 = \rho_0 g \int_z^{+\infty} \pi [R(\tilde{z})]^2 \, d\tilde{z}$$

(d) Deduce the first-order differential equation for $R(z)$:

$$\frac{dR}{dz} = -\frac{\rho_0 g}{2P_0} R(z).$$

Solution: First, switch the limits of integration to place z at the top boundary:

$$\pi P_0 [R(z)]^2 = -\rho_0 g \pi \int_{+\infty}^{z} [R(\tilde{z})]^2 \, d\tilde{z}.$$

Differentiating both sides with respect to z using the Chain Rule on the left and the Fundamental Theorem of Calculus on the right:

$$\pi P_0 \cdot 2R(z) \frac{dR}{dz} = -\rho_0 g \pi [R(z)]^2.$$

Dividing both sides by $2\pi P_0 R(z)$ (given $R(z) \neq 0$) simplifies directly to:

$$\frac{dR}{dz} = -\frac{\rho_0 g}{2P_0} R(z).$$

(e) Prove that the explicit solution for $R(z)$ given the initial base boundary condition $R(0) = R_0$ at ground level is :

$$R(z) = R_0 e^{-\frac{\rho_0 g}{2P_0} z}.$$

Solution: The differential equation $\frac{dR}{dz} = -kR(z)$ where $k = \frac{\rho_0 g}{2P_0}$ is a standard separable linear equation. Its general solution is:

$$R(z) = Ce^{-\frac{\rho_0 g}{2P_0} z}.$$

Applying the boundary baseline condition $R(0) = R_0$ gives $C = R_0$. Thus:

$$R(z) = R_0 e^{-\frac{\rho_0 g}{2P_0} z}.$$

(f) What famous tower does this structure remind you of ?

Solution: This profile is a continuous mathematical representation of the **Eiffel Tower** (Tour Eiffel) in Paris, whose geometric framework curves exponentially inward to distribute structural weight and wind loading perfectly uniformly down to the ground foundations.

11. (17 points) **Part 2 : vector fields through the tower.**

(a) Consider a horizontal cross-sectional slice of the tower $D(z)$ at a fixed height z. The boundary of this disk is a circle C_z of radius $R(z)$, oriented counterclockwise when viewed from above.

i. State Green's Theorem for a 2D vector field $\mathbf{F}(x, y) = L(x, y)\mathbf{i} + M(x, y)\mathbf{j}$ over the region $D(z)$.

Solution: Green's Theorem states:

$$\oint_{C_z} (L\,dx + M\,dy) = \iint_{D(z)} \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy$$

ii. Use Green's Theorem with the vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$ to compute the line integral $\oint_{C_z} \mathbf{F} \cdot d\mathbf{r}$, and show that it directly confirms the cross-sectional area formula $\text{Area}(D(z)) = \pi[R(z)]^2$.

Solution: For $\mathbf{F} = \langle -y, x \rangle$, we have $L = -y$ and $M = x$. Computing the partial derivatives:

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial L}{\partial y} = -1 \implies \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 1 - (-1) = 2$$

Applying Green's Theorem:

$$\oint_{C_z} \mathbf{F} \cdot d\mathbf{r} = \iint_{D(z)} 2\,dA = 2 \cdot \text{Area}(D(z)) = 2\pi[R(z)]^2$$

Dividing by 2 yields the standard area of the circular cross section: $\text{Area}(D(z)) = \pi[R(z)]^2$.

(b) Let E_h be the truncated 3D solid portion of the tower between the ground $z = 0$ and a finite height $z = h$. The boundary surface ∂E_h consists of three components: the base disk $D(0)$, the top disk $D(h)$, and the outer lateral surface S_L. Consider the 3D vector field $\mathbf{G}(x, y, z) = \langle x, y, 0 \rangle$.

i. Calculate the divergence $\vec{\nabla} \cdot \mathbf{G}$.

Solution: The divergence is $\vec{\nabla} \cdot \mathbf{G} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 1 + 1 + 0 = 2$.

ii. Set up the volume integral $\iiint_{E_h} \vec{\nabla} \cdot \mathbf{G}\, dV$ using the exponential profile $R(z) = R_0 e^{-\frac{\rho_0 g}{2P_0} z}$ and prove :

$$\iiint_{E_h} \vec{\nabla} \cdot \mathbf{G}\, dV = \frac{2\pi R_0^2 P_0}{\rho_0 g} \left(1 - e^{-\frac{\rho_0 g}{P_0} h}\right).$$

Solution:
The volume integral is:

$$\iiint_{E_h} 2\, dV = 2 \int_0^h \pi [R(z)]^2\, dz = 2\pi R_0^2 \int_0^h e^{-\frac{\rho_0 g}{P_0} z}\, dz.$$

Evaluating this single integral:

$$2\pi R_0^2 \left[-\frac{P_0}{\rho_0 g} e^{-\frac{\rho_0 g}{P_0} z} \right]_0^h = \frac{2\pi R_0^2 P_0}{\rho_0 g} \left(1 - e^{-\frac{\rho_0 g}{P_0} h}\right).$$

iii. Use the Divergence Theorem to compute the outward-pointing flux through the curved lateral surface S_L :

$$\iint_{S_L} \mathbf{G} \cdot \hat{\mathbf{n}}\, dS.$$

Solution: On the flat bottom disk $D(0)$, the outward unit normal is $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$. Since $\mathbf{G} \cdot (-\hat{\mathbf{k}}) = 0$, the flux is 0. On the flat top disk $D(h)$, the outward unit normal is $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$. Since $\mathbf{G} \cdot \hat{\mathbf{k}} = 0$, the flux is also 0.
By the Divergence Theorem:

$$\iiint_{E_h} \vec{\nabla} \cdot \mathbf{G}\, dV = \iint_{D(0)} \mathbf{G} \cdot \hat{\mathbf{n}}\, dS + \iint_{D(h)} \mathbf{G} \cdot \hat{\mathbf{n}}\, dS + \iint_{S_L} \mathbf{G} \cdot \hat{\mathbf{n}}\, dS.$$

Since the base and top fluxes are zero, the entire divergence flux escapes through the lateral wall:

$$\iint_{S_L} \mathbf{G} \cdot \hat{\mathbf{n}}\, dS = \frac{2\pi R_0^2 P_0}{\rho_0 g} \left(1 - e^{-\frac{\rho_0 g}{P_0} h}\right).$$

12. (15 points) **Part 3 : Tangent plane to the tower.**

(a) The outer boundary surface of the optimal tower can be described in Cartesian coordinates as the level surface of a function $F(x, y, z) = 0$, where:

$$F(x, y, z) = x^2 + y^2 - [R(z)]^2$$

i. Compute the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$, and state the gradient vector $\vec{\nabla} F(x, y, z)$ in terms of x, y, and the profile derivative $\frac{dR}{dz}$.

Solution: Taking the partial derivatives of $F(x, y, z) = x^2 + y^2 - [R(z)]^2$:

- $\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 - [R(z)]^2\right) = 2x$
- $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 - [R(z)]^2\right) = 2y$
- $\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 - [R(z)]^2\right) = -2R(z)\frac{dR}{dz}$

Thus, the gradient vector is:

$$\vec{\nabla} F(x, y, z) = \left\langle 2x,\, 2y,\, -2R(z)\frac{dR}{dz} \right\rangle.$$

ii. Substitute the known physical profile derivative $\frac{dR}{dz} = -\frac{\rho_0 g}{2P_0} R(z)$ into your gradient expression. Then, find the equation of the tangent plane to the tower surface at a specific point (x_0, y_0, z_0) lying on the boundary.

Solution: Substituting $\frac{dR}{dz} = -\frac{\rho_0 g}{2P_0} R(z)$ into the z-derivative yields:

$$\frac{\partial F}{\partial z} = -2R(z)\left(-\frac{\rho_0 g}{2P_0} R(z)\right) = \frac{\rho_0 g}{P_0}[R(z)]^2$$

Since the point (x_0, y_0, z_0) lies on the surface, $[R(z_0)]^2 = x_0^2 + y_0^2$. The normal vector $\mathbf{n}$ at this point is:

$$\mathbf{n} = \vec{\nabla} F(x_0, y_0, z_0) = \left\langle 2x_0,\, 2y_0,\, \frac{\rho_0 g}{P_0}(x_0^2 + y_0^2) \right\rangle.$$

The equation of the tangent plane is given by $\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$:

$$2x_0(x - x_0) + 2y_0(y - y_0) + \frac{\rho_0 g}{P_0}(x_0^2 + y_0^2)(z - z_0) = 0.$$

(b) At the top cap of the tower, the local height is given by the function function $f(x, y) = 4 - x^2 - y^2 + xy$. Find the critical point of $f(x, y)$ and use the Second Derivative Test to classify it (as a local maximum, local minimum, or saddle point).

Solution: To find the critical points, we compute the first partial derivatives of $f(x, y)$ and set them equal to zero:

$$f_x(x, y) = -2x + y = 0 \implies y = 2x.$$
$$f_y(x, y) = -2y + x = 0.$$

Substituting $y = 2x$ into the second equation yields $-2(2x) + x = -3x = 0$, which gives $x = 0$ and consequently $y = 0$. Thus, the unique critical point is at $(0, 0)$.

To classify this point, we apply the Second Derivative Test. We find the second-order partial derivatives:

$$f_{xx}(x, y) = -2, \quad f_{yy}(x, y) = -2, \quad f_{xy}(x, y) = 1.$$

We compute the discriminant (Hessian determinant) D:

$$D = f_{xx}(0, 0) f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

Since $D > 0$ and $f_{xx}(0, 0) = -2 < 0$, the Second Derivative Test confirms that the critical point $(0, 0)$ is a **local maximum**.

13. (10 points) Prove

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Hint : use the polar coordinates.

Solution: Let

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\ dx.$$

We have

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\ dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\ dx\right) \\ &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\ dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\ dy\right) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\ dxdy. \end{aligned}$$

By changing from the cartesians coordinates (x, y) to the polar coordinates (r, θ) :

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\ rdrd\theta \\ &= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{+\infty} \\ &= \pi\left[-e^{-\infty} + e^0\right] \\ &= \pi. \end{aligned}$$

Thus

$$I = \sqrt{\pi}.$$

Yet by symmetry of $x \mapsto e^{-x^2}$ we have :

$$I = 2\int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Finally

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Question	Points	Score
1	10	
2	3	
3	4	
4	3	
5	6	
6	8	
7	5	
8	5	
9	12	
10	22	
11	17	
12	15	
13	10	
Total:	120	